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GSM/M-20 **1614**

SPECIAL FUNCTIONS AND INTEGRAL TRANSFORMS

Paper–BM-242

Time Allowed : 3 Hours] [Maximum Marks : 40

Note : Attempt **five** questions in all, selecting at least **one** question from each Unit. Question No. **1** is compulsory. All questions carry equal marks.

Compulsory Question

1. (a) Show that $\sin x = 2[J_1 - J_3 + J_5 + ...]$. 2

(b) Express the following in terms of Legendre's polynomials $4x^3 - 2x^2 - 3x + 8$. 2

(c) Evaluate $\int_0^\infty t\, e^{-2t} \cos t \, dt$. 2

(d) If the Fourier transform of f(x) is $\bar{f}(s)$, then the Fourier transform of f(ax) is $\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$. 2

UNIT–I

2. (a) Solve $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ in series about $x = 0$. 4

(b) Prove that :

$$\frac{d}{dx}\left[I_n^2(x)+J_{n+1}^2(x)\right] = 2\left[\frac{n}{x}J_n^2(x)-\frac{n+1}{x}J_{n+1}^2(x)\right]$$

and hence show that

$$J_0^2(x) + 2\left[J_1^2(x)+J_2^2(x)+J_3^2(x)+......\right] = 1.$$ 4

3. (a) Find the solution of $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{1}{4}y = 0$ in terms of Bessel's function. 4

(b) Verify that the Bessel's function $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}$ sin x satisfies the Bessel's equation of order $\frac{1}{2}$. 4

UNIT–II

4. (a) Prove that :

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n.$$ 4

(b) Prove that :

$$\int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Hence deduce that :

$$\int_0^1 x^2 P_{n+1}(x) P_{n-1}(x)\, dx = \frac{n(n+1)}{(2n+3)(4n^2-1)}. \qquad 4$$

5. (a) Show that :

$$H_n(x) = 2^n \left[\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2} \right) \right] x^n. \qquad 4$$

(b) Prove that :

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} [H_n(x)]^2 dx = \sqrt{\pi}\, 2^n n! \left(n + \frac{1}{2} \right). \qquad 4$$

UNIT–III

6. (a) Find the Laplace transform of function $\sinh^3 2t$. 4

(b) Find :

$$L^{-1}\left[\frac{1}{s(s-6)^4} \right].$$

7. (a) Solve :

$$\int_0^t \frac{f(u)}{\sqrt{t-u}} = 1 + 2t - t^2. \qquad 4$$

(b) Solve the following equation by transform method:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}; \text{ where } y(0) = y'(0) = 1. \qquad 4$$

UNIT–IV

8. (a) Find the Fourier cosine transform of e^{-x^2}. 4

(b) Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$

Hence find $\int_0^\infty \frac{\sin x}{x} dx$. 4

9. (a) The initial temperature of an infinite bar is given by $\theta(x) = \begin{cases} \theta_0 & \text{for } |x|<a \\ 0 & \text{for } |x|>a \end{cases}$ determine the temperature at any point x and at any instant t. 4

(b) Using Parseval's identity prove that :

$$\int_0^\infty \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}.$$ 4